KARL FRITIOF SUNDMAN – THE MAN WHO SOLVED THE THREE-BODY PROBLEM

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1. Introduction

In the beginning of the 20th century an almost unknown Finnish mathematician-astronomer Karl Fritiof Sundman solved the three-body problem which one of the greatest mathematicians of that time Henri Poincaré had thought to be unsolvable. The problem itself was very simple – we know the coordinates of the three gravitationally bound bodies and their velocities at a certain moment and we have to find both the coordinates and velocities of these bodies at an arbitrary moment.

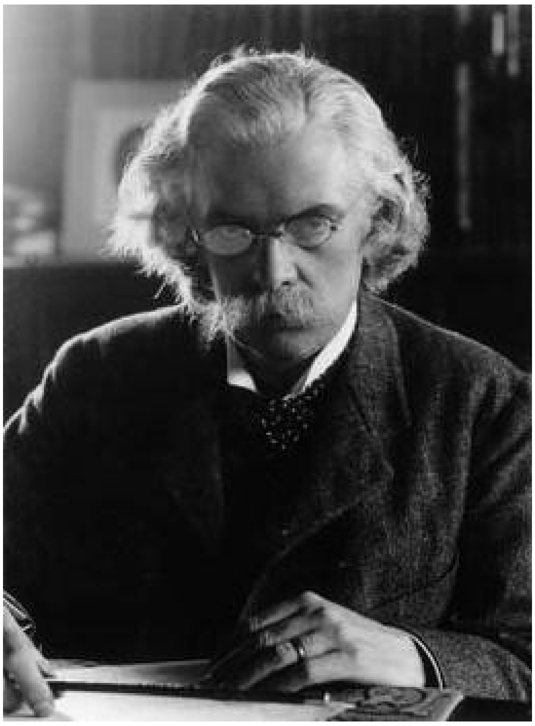
It is true that Sundman found the solution in the form of series which converges so slowly that one has to use $10^{8000000}$ members of that series in order to guarantee the accuracy acceptable at usual astronomical observations! This means that the solution has no practical value but principally the importance of it is overwhelming. And exactly that has been stressed lately when the theory of chaos has been rapidly developing.

2. Something about the state of mathematics and astronomy in Finland in the end of the 19th century and in the beginning the 20th century

In 1828 the Imperial Alexander University was founded in Helsinki – in the new capital of Finland - which actually was the successor of the Åbo Akademi founded already in 1640 and burnt down in the great Turku fire when approximately three quarters of the town perished in 1827. In Helsinki university there was a chair of mathematics through the 19^{th} and the beginning of the 20^{th} centuries which consisted of one professor and one or two assistant professors. Long time the professorship was hold by the alumnus of the same university Lorenz Lindelöf (1857-1874), who had been well recognized as a specialist in variational analysis but who had also publications in astronomy and he had twice held the position of the professor in astronomy. In 1877 the Swedish mathematician Gösta Mittag-Leffler was elected the successor of Lindelöf. It was a serious problem to get him working in the Finnish university because he was a Swede and thus did not know the Finnish language – to know the Finnish was a strict rule that applied to each and every lecturer. They had to apply for the exemption of that rule and Lindelöf fought as a lion for Mittag-Leffler.

So when at last the senat arranged the elections Mittag-Leffler wan scantily the other competitor Ernst Bonsdorff.

Mittag-Leffler was active in different areas of mathematics of which one may highlight the generalization of integral and differential operators. But mostly he will be remembered as the founder of the journal Acta Mathematica.



Magnus Gösta Mittag-Leffler

This journal was founded first of all by the financial support of king Oscar and Mittag-Leffler's wife Signe Lindfors. The journal published papers from the renowned mathematicians like Poincaré and Cantor in the first issues since Mittag-Leffler knew them personally. Up to this time the Finnish mathematicians had published their works in the journal *Acta Societatis Scientiarum Fennica (The proceedings of the Finnish scientific society)*, but this journal was still a provincial one which was unknown to the great mathematicians.

It cannot be left unnoticed that Mittag-Leffler had nurtured many a young mathematician like Hjalmar Mellin whom he sent to study at Weierstrass in Berlin. Only that enlarged the visibility of Finnish mathematics in the world to say nothing about the impact of Scandinavian (later the Northern countries) congresses of mathematicians which also took place under guidance of Mittag-Leffler.

In spite of the fact that Mittag-Leffler worked in Helsinki only four and a half years he remained tightly connected with Finland, especially with Lorenz Lindelöf. Edvard Neovius and Edvard Bonsdorff ran for the office of Mittag-Leffler, but since Bonsdorff withdrew himself then Neovius was elected the head of the chair (later his brothers changed their name to Nevanlinna). Neovius had studied in Zürich at Herman Amandus Schwarz, therefore under the influence of his teacher he started to investigate minimal surfaces.

Hugo Gyldén

Neovius left the post for the head of the Finnish financial department in 1900 and the chair had to find a new head.

In the same year when Neovius was elected the head of the chair of mathematics, Anders Donner was elected the head of the chair of astronomy and the director of the observatory. He had been educated in Helsinki but later he had been educating himself at Weierstrass in Berlin and he had written the doctoral thesis about elliptical functions. Before the election he had been working with Hugo Gyldén in Stockholm, where he studied the theory of perturbations.

After Neovius Ernst Leonard Lindelöf (a son of Lorenz) was elected the head of the chair of mathematics. He had written his thesis about the Lie transformation groups. He had studied in different European universities, e.g. in Stockholm, Paris and Göttingen, thus entering into relations with Mittag-Leffler, Paul Painlevé and Emil Borel. His works in the field of theory of functions were so important that in 1907 he was elected a member of the editorial board of *Acta Mathematica*.

Hugo Gyldén, who was mainly dealing with celestial mechanics, could only be conditionally named a Finnish astronomer because he studied in Helsinki and was a short time the assistant professor of astronomy in the Helsinki university but his main bulk of scientific work was done in Stockholm. His thesis were written in Gotha, Germany at then the best celestial mechanic Peter Hansen, though he had investigated very successfully the refraction in the Earth's atmosphere at Pulkovo.

Karl Fritiof Sundman

Yet his influence on the Finnish astronomy was great, even for the fact that he had been the teacher of Anders Donner, to say nothing about his substantial help in publishing Sundman's papers.

3. Biography in a nutshell

Karl Fritiof Sundman was born on October 28, 1873 in a small Finnish town Kaskinen (Kaskö) on the shore of the Bay of Bothnia in the family of a customs officer. His parents – father Johan Fritiof Sundman and mother Adolfina Fredrika Rosenqvist – wished him to become a fisherman. However, this was not to be the case since Karl Fritiof was not suited for such an occupation. Therefore, his parents were seriously concerned about his future. But the boy was interested in studying and when he - mainly as a self-taught person – had reached the necessary educational level he entered the Imperial Alexander university in Helsinki to study exact sciences. In addition to that he worked as an assistent in the university observatory in the department of astrophotography during 1894-1897. And on May 8th, 1897 he was promoted to candidate of philosophy. But this scientific degree did not last long since on May 31st he got the degree of master of philosophy (with the qualificatory notice „ultimus“. The best notice would have been „primus“, „ultimus“ was the second best). In 1897 to 1899 he continued his studies in Pulkovo observatory as an assistant to Oskar Backlund who was the director of the observatory at that time. Sundman helped him in finishing the monumental work of Hugo Gyldén about the perturbations of the orbits of the big planets. At the same time he was writing his thesis under the guidance of Backlund. In 1899 Sundman returned to Helsinki university. He passed the licentiate exam on May 29th, 1901 having before that submitted his dissertation *"Über die störungen der kleinen Planeten, speciel derjenigen, deren mittlere Bewegung annähernd das doppelte Jupiters beträgt".* In that work he set as a goal to ascertain the perturbations of small planets, especially of those planets whose orbits compared with the Jovian orbit were specific but his method could have been used also for other planets, too. Swedish astronomer Edvard Hugo von Zeipel described the Sundman method as outstanding. Sundman was given the doctor's degree on April 22nd, 1903 without the defence but already before that on March 1903 he was elected the assistant professor of astronomy in the same Imperial Alexander university in Helsinki.

The following years, 1903-1906, he spent researching some astronomical problems as a stipendiat of Rosenberg in Göttingen, Paris, München, Leipzig and

Berlin (Rosenberg travel grant was founded by Vaasa county governor Herman Rosenberg's extramarital son H.F. Antell in memory of his father). On November 14, 1907 Sundman was named the extraordinary professor of astronomy and on January 5, 1918 he was named the ordinary professor of astronomy and also the director of the university observatory, now already in independent Finland. Actually he had been the acting professor of astronomy since professor Anders Donner had been elected the rector of the university. Since Donner had become the chancellor of the university in 1915 the posts of the head of the chair of astronomy and the director of the observatory had become vacant. Three candidates applied to that post: Karl F. Sundman, Ilmari Bonsdorff and Ragnar Furuhjelm. The last two were observers and Sundman, surely, theoretician. Sundman was elected but the two others were not satisfied with the result and they filed in a complaint. The problem was a political one – what kind of a leader was to be preferred, an observer or a theoretician. The election commission solved the problem rather elegantly – Sundman remained on the posts, for Furuhjelm a new chair was founded and Bonsdorff became a director of a newly founded institute of geodesy. Sundman was the director of observatory and the professor of astronomy in Helsinki university up to May 31, 1941. He died on September 28, 1949 after a very long illness.

4. Sundman's travels

As already said Sundman travelled to the scientific centres of western Europe on the financial aid of Rosenberg grant. The travel was planned for dealing the problem *"The investigation of the motion of small planets and their moons, especially Saturn's moons, in order to treat the case of near commensurability between the mean motions of the perturbing and perturbed bodies".*

As Sundman himself later claimed he had got interested in that problem because of the works by Gyldén on the research of the stability of the planetary system. Gyldén had used for it elliptical functions but not successfully. Sundman hoped to find help from using the ordinary trigonometric functions. Shrewdly he added the idea of using the theory of functions into his application. This hint was not left without attention from the member of the respective university commission, Sundman's colleague Ernst Lindelöf, who was known as a specialist in the theory of functions!

Sundman hoped to meet Henri Poincaré, Paul Appel, Paul Painlevé and Octave Callandreau in Paris, and Hugo von Seeliger in Munich. Though Sundman indeed met many mathematicians and astronomers during his travel, only Poincaré and Gyldén were mentioned in his report. As far as he corresponded with Donner one can find much more information in his letters than in the official report. For instance, he had a talk with Poincaré about some simple cases of the three-body problem with possible singularities.

On April 13, 1904 Sundman was elected a member of *Société Mathématique de France* while he was presented by Painlevé and Borel. Sundman's biographer Barrow-Green thinks that Borel's presentation came through Lindelöf because these two men were good friends.

In Göttingen Sundman met Karl Schwarzschild, who recommended him to write a chapter of astronomy in the Klein's monumental book *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihre Anwendungen*, where Schwarzschild and Oppenheim were the editors of the astronomical part.

In Munich Sundman had a useful talk with von Seeliger and in Leipzig he met Heinrich Bruns, who was then the professor of astronomy and the director of the observatory and who had published an essential paper about the three-body problem.

At the end of April 1905 Sundman went to Padua in order to meet Levi-Civita, though this journey was not in his schedule. The aim of this journey was to discuss the three-body problem with this famous mathematician during two weeks.

5. The history of the three-body problem

As already told the three-body problem is very simple: we have three gravitationally bound bodies and we know their coordinates and velocities at a certain moment. We have to find their coordinates and velocities at some other time. Of course, mathematically are these bodies only masspoints but even such an approach is a rather good approximation to a real situation in astronomy where the bodies have both masses and dimensions.

In its traditional sense the problem got its beginning in 1687 when Newton published his *Principia*. The next famous man who dealt with the three-body problem was Leonard Euler. He treated the simplified problem when the mass of one of the bodies is negligible compared with two others. Later Poincaré named this case the restricted three-body problem. Almost at the same time the three-body problem was dealt with by Joseph Louis Lagrange, who solved many special cases of which one consisted of three bodies situated at the vortices of an equilateral triangle. In this case the configuration is stable. This kind of configuration is realized in the solar system, namely two groups of asteroids are moving in the orbit of Jupiter, but one group is ahead of Jupiter 60 degrees and another is behind Jupiter 60 degrees.

Carl Jacobi and Henri Poincaré dealt with the three-body problem very thoroughly, especially Poincaré. But neither of them could find the general solution. Ernst Bruns and Henri Poincaré proved instead that this problem lacks a general analytic solution which could be expressed algebraically or in integral form. Generally the gravitationally bound three-body movement is non-periodic.

But the search for a solution continued despite of the idea of Poincaré that the solution of that problem seemed to be very far in the future.

In 1859 the Swedish king Oscar II, who himself had studied mathematics, announced on the occasion of his 60th birthday in 1859 a contest of essays for four problems in the theory of functions of which one concerned the movement of optional number of gravitationally bound bodies. The prize commission (Mittag-Leffler, Hermite ja Weierstrass), found that Poincaré, who had treated the restricted three-body problem, was the winner and got as a prize a gold medal and 2500 Swedish kronas. However, he had not found the general solution. Yet he had opened the door to the theory of dynamical systems. According to the prize ordinance the winner had to publish his work in journal *Acta Mathematica*. During the editorial work one of co-editors of the journal E. Phragmén asked the author a question, after which it became clear that a big part of Poincaré's work was wrong. Certainly the author corrected his work and discovered something new in doing it, namely isoclinic points, but then the journal issue was already out of press and sent to subscribers. To avoid a scandal Mittag-Leffler decided to destroy all the edition and print it anew. Poincaré had to pay for it - 3585 kronas and 63 öre (it should be said as an explanation that Mittag-Leffler's annual salary was 7000 kronas in 1881).

Mathematically this problem does not seem to be complicated – we have nine second-order ordinary differential equations which could be described by 18 first-order ordinary differential equations using the hamiltonian. The solution of this system is made difficult by the fact that we have only 10 algebraic independent integrals: 6 integrals for the movement of the mass centre, 3 integrals for the angular momentum and the energy integral. Using these integrals the system could be simplified to one sixth order ordinary differential equation but not more.

When mathematicians understood that there was no analytically general solution to that problem they started to seek a solution as infinite series (using the so-called Frobenius method). At last this method was successful but it took a very long time because the impacts of the bodies were difficult to describe mathematically. True enough, the mathematicians knew that the impacts could be described mathematically as singularities in the differential equations. But it was not clear how to get rid of them. Moreover, it was unclear in the beginning whether there were singularities other than impacts.

In 1886 Poincaré found that if we knew beforehand that the distances between any two bodies are larger than some given limit, then it is possible to find the coordinates of the three bodies as convergent series for any time moment. But the trouble was in finding this limit.

In 1895 Painlevé showed that the only singularities in the three-body problem are the impacts and the problem is solvable provided we know that the initial conditions of equations exclude any impacts between these bodies. Sure enough, the mathematicians started to look for these initial conditions and Levi-Civita found the Painlevé solution for the restricted three-body problem. Another italian mathematician Giulio Bisconcini found for the general three-body problem two analytical formulas connecting the initial conditions which allowed to determine the singular trajectory, i.e. during a finite time of an impact. However this did not solve the general problem.

Here I would like to add a reference to a paper by Martin C. Gutzwiller about the oldest three-body problem in astronomy which, as far as we know, got its beginning ca 3000 years ago in Mesopotamia, namely about the system Moon-Earth-Sun. In this paper one finds a thorough review of the theories of the Moon movement and their development through ages.

6. The solution of Sundman

Sundman arrived at the three-body problem through the study of the stability of the Solar system. In her thorough paper Barrow-Green analyses the increase of Sundman's interest in this problem and how and where he studied this mathematics that was needed to solve the problem. Evidently we have to start from the fact that Sundman was a gifted mathematician already in the beginning of his career. And this fact that he had to chew through Gyldén's *Magnum opus*, when he worked in Pulkovo, helped to make his knowledge of mathematics even better.

When Sundman was again in Helsinki then without doubt his friendship with Ernst Lindelöf – the best specialist of the theory of complex functions in Finland and three years older than Sundman – was of great help. Sundman himself admitted it in his paper of 1909 where he expressed thanks to Lindelöf for simplifying many proofs in his work. Sundman profited also from Lindelöf's abundant acquaintances in Sweden, Germany and France.

His first paper on the three-body problem Sundman published in the proceedings of the Finnish scientific society in 1907. Two years later another his paper followed in the same journal and in 1909, too, he gave a presentation on the three-body problem on the conference of Scandinavian mathematicians. These two papers were later joined in a Mittag-Leffler's journal *Acta Mathematica* in 1912.

In the Sundman's solution it is important that he considered not the two-body impacts as it was done usually but promptly the impacts of three bodies, e.g. the emerging singularities. In doing this he formulated a very important theorem needed to obtain the solution that if all three integrals of angular momentum are zero, only then the impacts of three bodies can happen. In the case when not all three integrals are zero one can prove that there exists a positive quantity of which the biggest distance between the three bodies cannot be smaller. In addition to that Sundman ascertained that if two bodies are impacting then the angular velocity of one around the other is finite.

In his paper of 1909 he proved that when not all three angular momenta are equal to zero then one can find such a variable τ (by the way, following Poincaré) that the coordinates of the bodies, the distances between them and the time can be shown to be holomorphic functions, or, to be exact, the convergent exponential series which describe the movement of the bodies for all real time values, irrespectively of the types of impacts.

This statement contains an essential info – that after the impact the solution can be analytically continued and that by substitution of a variable one can eliminate the singularity at the impact of two bodies. There was another smart move in the proof – Sundman defined the variable τ on the complex plane. Though one could not attach any physical meaning to that generalization then despite of that it was a very important step towards the whole solution.

What concerns the publishing of Sundman papers then Lindelöf and Mittag-Leffler helped him a great deal, especially in publishing the third, concise paper in the Mittag-Leffler's journal Acta *Mathematica*. This paper was decisive in getting the Pontecoulant prize of the French academy of sciences. And as the Sundman paper was exceptionally important the academy doubled the usual prize.

We have already told that Sundman published his first papers on the problem in the proceedings of the Finnish scientific society in French language, though, but in an internationally almost unknown journal. Whereas it was so we learn from the speech of the president of the British association of mathematics Sir George Greenhill in 1914 where he said that „The dramatic episode of Sundmann (!) in Helsingfors should encourage the young mathematician as exhibiting the inexhaustible nature of our subject ...But as Sundmann's memoir was published in the Finnish language of Helsingfors, the copy sent to Poincaré remained on his desk unread, and Sundmann's name is not mentioned in the Presidential Address at the Cambridge Congress of Mathematicians.“ Evidently Greenhill had not seen these papers though their copies were present both in Cambridge and in London. But we may believe that Poincaré knew these papers by Sundman more so since Sundman met him in Paris. Eric T. Bell writes in his famous book "Men of Mathematics" about one interesting episode in connection with Poincaré and Sundman (Sundman's name is not mentioned in this episode but without any doubt we can be sure that this man was Sundman). Some mathematician had come from Finland to meet Poincaré and talk about scientific matters. Though the maid had informed Poincaré that the guest had arrived and was waiting in the adjoining room Poincaré continued pacing back and forth behind the flimsy curtains for three solid hours. At last he thrusted his head through the curtains and roared: "Vous me dérangez beaucoup!" Sundman stood up in silence and departed. This scene characterizes well both men – one was very absent-minded when he thought about scientific problems and the other was extremely modest.

6. How one regarded the Sundman solution?

Barrow-Green divided the consideration of the Sundman solution into three periods.

During the first period – directly after the first two publications in the proceedings of the Finnish scientific society – nobody bestowed any consideration upon them. Mainly because both the author and the journal were unknown to the international brotherhood of mathematicians.

The second period started with the publication of the Sundman solution in *Acta Mathematica*. This journal was already known since the Editor-in-Chief Mittag-Leffler who used every opportunity to promote the Scandinavian and Finnish mathematics, had invited many illustrious mathematicians to publish in this journal. As a result the Sundman solution was highly praised. Most certainly the mathematicians understood that in practical calculations the solution was useless but the very substantial problem in mathematics and astronomy that had been unsolvable for so many celebrities, had been solved.

The third period which started after the I WW and lasted up to the beginning of the 90-s, marked the diminishing of interest towards the classical celestial mechanics. It appeared as if Poincaré (mainly) and Sundman, too, had taken this science to such heights that nobody wanted to deal with it any more. True enough, the space flights helped to reanimate the practical problems of the celestial mechanics.

The fourth period which started in the beginning of 90-s, marked the growth of interest because both the theory of chaos and the *N*-body problem in cosmology were developing fast and so gave an impetus to the investigation of the Sundman solution. This in turn meant that the work of Sundman became actual.

This period continues with very deep results obtained by two physicists in Belgrade university - Milovan Šuvakov and Veljko Dmitrašinović, who tried to find periodic trajectories in the movement of the three bodies. They found 13 new families of this kind of movement (three families were known already before). The earliest periodic orbits were found by Lagrange and Euler, but only in 1970-s USA mathematician Broucke and French mathematician Hénon found with the help of computers some new orbits. So only three families were known before: Lagrange-Euler, Broucke-Hénon and the figure-eight family (discovered by the american physicist Moore in 1993 and this represents three bodies chasing each other in an orbit in the form of number eight). The solution of Lagrange-Euler is simpler: three bodies moving along a circle and separated from each other in 120 degrees. Broucke-Hénon solution considers the two bodies moving to and fro along a line and the third body orbits them.

Milovan Šuvakov and Veljko Dmitrašinović took into use a completely new classification of the families, namely an abstract space called the shape-sphere which describes the orbits in relative units of the distances between the bodies. For instance the Lagrange-Euler family depicts only a point on the shape-sphere and at the same time the family „ball of yarn“ looks indeed as a ball of yarn on the shape-sphere after a kitten has played with it.

7. Sundman's other works

Before his paper on the three-body problem in 1907 Sundman had published very little. In addition to his doctoral thesis he was the author of three papers only. One of them was on the continued fractions, another concerned the ring micrometer and the third dealt with the determination of the Gyldén *A*- and *B*- coefficients.

Here we have to stress Sundman's work in Pulkovo where he helped Backlund to finish the Gyldén work on the planetary orbits. When Sundman left Pulkovo for Helsinki then Backlund had to spend many years to finish this work and though this work was published under Backlund's name in 1908, a lot of Sundman's efforts went into it.

We have already mentioned Sundman's paper on the theory of planetary movement in Klein's Encyklopädie in 1915. According to Carl Källman's info Karl Schwarzschild proposed Sundman to write this article for the encyclopedia already in the beginning of March, 1904. This means that Schwarzschild had to be of a very good opinion of Sundman. Sundman wrote this paper almost eight years, because he included his own newest results in perturbations of the planetary orbits in it. However, it is very strange that in his very long paper – longer than any other paper on mathematics in this encyclopedia – he did not deal with the general dynamical principles which are largely used in celestial mechanics. He left out the theories of the Moon movement and the three-body problem, too.

In the same 1915 he published yet another paper. Since the computing of the planetary coordinates is long and bothersome Sundman decided to build a mechanical computer to determine orbits of asteroids and he named it „the perturbographe“ (un méchanisme imite les mouvements de la planète perturbatrice et de la planète perturbée de telle façon que leurs positions relatives dans l'espace sont indiquées à une échelle donnée par les positions relatives de certaines organes de la machine). The main goal of it was to compute the perturbations in the movement of asteroids caused by Jupiter. In essence, the perturbographe had to solve the differential equations of the second order. Sundman managed to present the blueprints of the perturbographe. The computer had to be accurate and quick – it took seven minutes to compute the perturbations of a planet while a human spent for the same work with the same accuracy many weeks. We do not know whether the perturbographe was ever built or not, though enterprizing Mittag-Leffler showed big interest in it.

Sundman regarded the theory of general relativity of Einstein with big caution. When Lindelöf got 60 in 1929 Sundman published the paper *La gravitation universelle et sa vitesse de propagation* in the proceedings of the Finnish scientific society. In this paper he showed that the shift of the perihelion of Mercury could be explained on the basis of the classical mechanics provided we use some artificial helping hypotheses. One of them was a hypothesis of the gravitation moving with the velocity of light. For Sundman space remained always euclidian and time absolute.

When Svante Elis Strömgren got 70 in 1940, Sundman published a paper on an original proof of a Poisson theorem in his honour.

Even after leaving the professorship in 1941 Sundman continued his scientific work. He computed for the 1945 solar eclipse the movement of the Moon with exceptional accuracy using only the 10 decimal place table of logarithms (the paper about it was published in the proceedings of the Baltic geodetic commission in 1948). The director of the Finnish institute of geodesy Bonsdorff proposed an idea to use these calculations for determination of the distance between the Eurasian and American continents. The tests in 1945 were successful and during the eclipse in 1947 the Bonsdorff idea was put into practice. The distance between Ghana and Brasil was determined with the accuracy of 140 meters.

8. Conclusion

Despite of his big renown Karl Sundman remained very modest as a human being. His kindhearted and attractive nature helped him to find many friends both in the midst of his colleagues and outside of it but he made everything to be left out of the limelight.

His lectures were matchlessly businesslike and he never aspired to make them elegant. When he was the director of the Helsinki university observatory he regarded as the main goal of the observatory – following his predecessor Anders Donner – to compile a big photographic stellar catalogue. In his directorship, too, he remained reserved, during his time of more essential instruments only the Krüss microdensitometer was purchased.

Sundman regarded the meetings of the Finnish scientific society very relevant and he missed them very seldom. He was also the foreign member of the Royal Swedish academy of sciences and the member of the editorial board of the journal *Acta Mathematica*.

One is sure - Karl Fritiof Sundman remains forever in the annals of mathematical astronomy as a man who solved the three-body problem.

Acknowledgments

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