# TEMPERATURE DISTRIBUTION IN A SEMI-INFINITE ATMOSPHERE SUBJECTED TO COSINE VARYING COLLIMATED RADIATION

Tõnu Viik

Tartu Observatoorium (Tartu Observatory), 61602 Tōravere, Tartumaa, Eesti (Estonia)

Published in the Proceedings of the Estonian Academy of Sciences, Phys.-Math. Series vol 49, 40-57, 2000.

**Abstract** - Accurate numerical solutions are presented for the radiation field in a semi-infinite, two-dimensional, plane-parallel, absorbing-emitting but non-scattering gray atmosphere subjected to cosine-varying collimated incident boundary radiation. We approximate the kernel of the integral equation for the emissive power by a sum of exponents. After this approximation the integral equation can be solved exactly. The solution contains the well-known Ambartsumian-Chandrasekhar $H-$function. Some methods to determine this function are considered in detail.

This approach allowed us to find the accurate values for the emissive power and the radiative flux at arbitrary optical depths in the atmosphere. The calculations have shown that the radiative flux may have a maximum at certain values of the spatial frequency in the atmosphere and that the region, where the emissive power reaches a constant value, may lie very deep in the atmosphere.

**Key words**: 2D radiation transfer, $H$-function, emissive power, radiative flux

# 1 INTRODUCTION

The one-dimensional model for radiative transfer in different media has been extensively studied. Many problems in one-dimensional transfer allow rigorous mathematical solutions which may serve as benchmarks for more complicated cases or as first approximations to two-dimensional problems.

However, there are many problems in astrophysics, meteorology, fluid mechanics, gas dynamics, energy transfer between surfaces where the results of the one-dimensional model of radiative transfer are not accurate enough and we have to apply models of two- or three-dimensional radiative transfer. This enormously complicates the solution and thus only a few exact studies exist and they deal mostly with scattering of a narrow pencil of radiation incident on a scattering medium [1–5].

There is a group of two-dimensional problems, though, for which an exact solution can be found. These problems are connected with non-scattering media with the following types of boundary radiation: (1) cosine varying collimated radiation, (2) strip of collimated radiation, (3) cosine varying diffuse radiation, and (4) constant temperature strip. In these cases the two-dimensional problem can be reduced to one-dimensional integral equations by the method of separation of variables. These problems are considered in a series of papers by Breig and Crosbie [6–11] where also a good review of literature on the subject is given. Their approach allowed to determine only the external radiation field.

Mueller and Crosbie [12] have carried the investigation further by considering three-dimensional radiative transfer with polarization and multiple scattering. This paper gives a very good review of the latest studies in multi-dimensional transfer.

In this paper we try to generalize the results of Breig and Crosbie by applying the method of approximating the kernel of the integral equation for the Sobolev resolvent function (which essentially is the regular part of the respective Green function) by a series of exponents. The resulting approximate equation has an exact solution which is also represented by a series of exponents. This allows us to define the auxiliary functions $g$ and $h$ through the resolvent function $\Phi$ and thus define all the relevant functions.

In one-dimensional media the described approach gave very accurate results [13]. Though for the problem at hand the characteristic function of radiative transfer is not an even polynomial as in the case of one-dimensional transfer, it still has retained an essential feature - its evenness in angular variable. This allowed us to expect accurate results also for the problem under investigation. It appeared that this was really the case and we were able to find both the external and internal radiation field in a simple and concise way for a semi-infinite, two-dimensional, plane-parallel, absorbing-emitting but non-scattering gray atmosphere subjected to cosine-varying collimated incident radiation.

# 2 SOLUTION OF THE EQUATION OF RADIATIVE TRANSFER

We are looking for the emissive power  in a homogeneous non-scattering plane-parallel two-dimensional gray atmosphere which is in local thermodynamic equilibrium. The radiative transfer in such an atmosphere is described by the following equation

$$\cos\theta\frac{\partial I}{\partial\tau_z}+\sin\theta\sin\phi\frac{\partial I}{\partial\tau_y}+I=\frac{\sigma}{\pi}T^4, \tag{1}$$

where $I$ is the intensity; $\theta$, the polar angle measured from the inward normal to the atmosphere; $\phi$,azimuthal angle measured from the $\tau_x-$axis; $\sigma$, the Stefan-Boltzmann constant; $T$, the temperature in the atmosphere; $\sigma T^4$,the emissive power. The optical depth $\tau_z$ is measured downward from the boundary of the atmosphere and together with $\tau_x$ and $\tau_y$ they form a right-hand rectangular co-ordinate system. We require that the energy is transferred only by radiation, i.e. there is no heat conduction or convection in the atmosphere.

Applying of integrating factor techniques to Eq. (1) we obtain the formal solution for the intensities of downward and upward moving radiation in the form

$$I^+(\tau_y,\tau_z,\mu)=I_0(\tau_y^+)\exp(-\tau_z/\mu)+\frac{1}{\pi}\int_0^{\tau_z}\sigma T^4(\tau_y',\tau_z')\exp(-(\tau_z-\tau_z')/\mu)\mathrm{d}\tau_z'/\mu \tag{2}$$

and

$$I^-(\tau_y,\tau_z,\mu)=\frac{1}{\pi}\int_{\tau_z}^{\infty}\sigma T^4(\tau_y',\tau_z')\exp(-(\tau_z'-\tau_z)/\mu)\mathrm{d}\tau_z'/\mu, \tag{3}$$

where $\tau_y^+=\tau_y-\tau_z\tan\theta\sin\phi$, $\tau_y'=\tau_y+(\tau_z'-\tau_z)\tan\theta\sin\phi$, $\mu=\cos\theta$ and $I_0^+$ is the intensity incident on the boundary of the atmosphere [1].

As far as we require the atmosphere to be in radiative equilibrium we may write

$$4\sigma T^4(\tau_y,\tau_z)=\int_{4\pi}I\mathrm{d}\omega, \tag{4}$$

where $\omega$ is the solid angle.

Substituting Eqs. (2) and (3) into Eq. (4) we obtain the equation for the emissive power

$$4\sigma T^4(\tau_y,\tau_z)=\int_{2\pi}I_0^+(\tau_y^+)\exp(-\tau_z/\mu)\mathrm{d}\omega+$$
$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_0^{\infty}\sigma T^4(\tau_y',\tau_z')\exp(-\left|\tau_z-\tau_z'\right|/\mu)\mathrm{d}\tau_z'\mathrm{d}\mu/\mu\mathrm{d}\phi. \tag{5}$$

According to our assumption the incident intensity may be expressed as

$$I_0^+(\tau_y^+) = I_0\left[1 + \epsilon \exp(i\beta\tau_y^+)\right]\delta(\mu - \mu_0)\delta(\phi), \tag{6}$$

where $I_0$ is a constant, ($\mu_0 = \cos\theta_0, \phi$) defines the direction of the incident collimated radiation, $\epsilon$ is the amplitude of the cosine wave and $\delta$ is the Dirac delta function. Boundary condition (6) means that the top of the atmosphere is illuminated strip-wise by a parallel beam at an angle $\theta_0$ while the strips are parallel to the $x$ - axis and their widths are defined by the spatial frequency $\beta$ as $\pi/\beta$ in units of optical length $\tau_y^+$. The illumination in the direction parallel to the $y$ - axis varies according to the cosine law. Next we apply the concept of separation of variables to Eq. (5) by assuming that

$$\sigma T^4(\tau_y, \tau_z) = \frac{1}{4} I_0\left[B_{\beta=0}(\tau, \mu_0) + \epsilon B_\beta(\tau, \mu_0)\exp(i\beta\tau_y)\right], \tag{7}$$

where $B_\beta$ is the dimensionless emissive power and $\tau = \tau_z$. Using Eq. (7) in Eq. (5) gives us a simple integral equation for $B_\beta$ in the form

$$B_\beta(\tau, \mu_0) = \exp(-\tau/\mu_0) + \frac{1}{2}\int_0^\infty \mathcal{E}_1(\tau - \tau')B_\beta(\tau', \mu_0)\mathrm{d}\tau', \tag{8}$$

where the generalized exponential integral $\mathcal{E}_1$ is defined as [2]

$$\mathcal{E}_1(\tau, \beta) = \int_1^\infty \exp(-|\tau|\sqrt{t^2 + \beta^2})\frac{\mathrm{d}t}{\sqrt{t^2 + \beta^2}}. \tag{9}$$

By substituting $\mu_0$ for $(t^2 + \beta^2)^{-1/2}$ in Eq. (8) and multiplying both sides of it by $\mathrm{d}t/\sqrt{t^2 + \beta^2}$, and last, integrating from 1 to $\infty$, we arrive at the integral equation for the resolvent function $\Phi_\beta$ in the form

$$\Phi_\beta(\tau) = \frac{1}{2}\mathcal{E}_1(\tau, \beta) + \frac{1}{2}\int_0^\infty \mathcal{E}_1(\tau - \tau')\Phi_\beta(\tau')\mathrm{d}\tau', \tag{10}$$

where

$$\Phi_\beta(\tau) = \frac{1}{2}\int_1^\infty \frac{B_\beta\left(\tau, \sqrt{t^2 + \beta^2}\right)\mathrm{d}t}{\sqrt{t^2 + \beta^2}}. \tag{11}$$

Next we introduce two functions $h(\tau, \mu)$ and $g(\tau, \mu)$ as follows [13]

$$h_\beta(\tau, \mu) = 1 + \int_\tau^\infty \Phi_\beta(t)\exp(-(t - \tau)/\mu)\mathrm{d}t \tag{12}$$

and

$$g_\beta(\tau,\mu) = \exp(-\tau/\mu) + \int_0^\tau \Phi_\beta(t)\exp(-(\tau-t)/\mu)\mathrm{d}t. \qquad (13)$$

In the following we need a system of equations which connect those two functions with each other

$$-\mu\frac{\partial h_\beta(\tau,\mu)}{\partial\tau} + h_\beta(\tau,\mu) = \mu\Phi_\beta(\tau) + 1, \qquad (14)$$

$$\mu\frac{\partial g_\beta(\tau,\mu)}{\partial\tau} + g_\beta(\tau,\mu) = \mu\Phi_\beta(\tau). \qquad (15)$$

Eqs. (14) and (15) can easily be found from Eqs. (12) and (13) by differentiating them with respect to $\tau$.

Sobolev [18] has shown that the solution of Eq. (8) may be written in the form

$$B_\beta(\tau,\mu_0) = B_\beta(0,\mu_0)\left[\exp(-\tau/\mu_0) + \int_0^\tau \Phi_\beta(t)\exp(-(\tau-t)/\mu_0)\mathrm{d}t\right], \quad (16)$$

or, in our notation,

$$B_\beta(\tau,\mu_0) = B_\beta(0,\mu_0)g_\beta(\tau,\mu_0) \qquad (17)$$

Formally this completes the solution of the problem to determine the temperature distribution in a semi-infinite atmosphere subjected to collimated cosine varying radiation.

Next we show how to find the emissive power at the boundary $B_\beta(0,\mu_0)$ and the function $g_\beta(\tau,\mu)$ at an arbitrary optical depth.

It is obvious that if $\beta = 0$ then Eq. (8) reduces to the equation describing radiation transfer in an one-dimensional medium which have been successfully solved by introducing the Sobolev's resolvent function [2] and then approximating it by a sum of exponents. Since Eq. (8) is linear and the kernel is a sum of exponents we may try to use the same technique.

First we change the variable $u = (t^2 + \beta^2)^{-1/2}$ in Eq. (9) to reduce this formula to a more familiar form

$$\mathcal{E}_1(\tau, \beta) = \int_0^p \frac{\exp(-\tau/u)}{\sqrt{1 - \beta^2 u^2}} \frac{\mathrm{d}u}{u}, \tag{18}$$

where $p = (1 + \beta^2)^{-1/2}$.

To solve Eq. (10) we express the generalized exponent integral in Eq. (18) as a sum of exponents

$$\mathcal{E}_1(\tau, \beta) = 2 \sum_{k=1}^{N} w_k \mu_k^{-1} \Psi_k \exp(-\tau/\mu_k), \tag{19}$$

where the characteristic function is expressed as

$$\Psi_k = \frac{1}{2\sqrt{1 - \beta^2 \mu_k^2}}. \tag{20}$$

In Eq. (19) $w_i$ and $\mu_i$ are the weights and points of a Gauss quadrature rule in the interval $(0, p)$ and $N$ is the order of the quadrature [3]. The characteristic function $\Psi$ (different from that which appears in the analysis by Breig and Crosbie [7] but nevertheless giving accurate results!) is not a polynomial but it has retained another important quality - it still is an even function of $x$.

If we have approximated the general exponential integral as a sum of exponents then Eq. (10) accepts an exact solution as a sum of exponents [13]

$$\Phi_\beta(\tau) = \sum_{i=1}^{N} a_i \exp(-s_i \tau). \tag{21}$$

In order to determine the coefficients $a_i$ and $s_i$ in Eq. (21) we use Eq. (21) in Eq. (10) and by equating the similar exponents we obtain the characteristic equation

$$1 - 2 \sum_{i=1}^{N} \frac{w_i \Psi(\mu_i, \beta)}{1 - \mu_i^2 s^2} = 0, \tag{22}$$

and a linear algebraic system for coefficients $a_i$

$$\sum_{k=1}^{N} \frac{a_k}{1 - \mu_i s_k} - \mu_i^{-1} = 0, \qquad i = 1, ..., N. \tag{23}$$

It is evident that Eq. (22) has exactly $N$ pairs of non-zero solutions $\pm s_k$ if only $\beta \neq 0$. If $\beta = 0$ then $s_1 = \pm 0$ is also a solution but as far as this takes us back to the thoroughly studied one-dimensional case, we shall not consider it here.

The roots of the characteristic equation satisfy the following inequalities

$$0 \leq |s_1| < \mu_N^{-1} < |s_2| < \mu_{N-1}^{-1} < ... < |s_N| < \mu_1^{-1}.$$

As far as the roots are bracketed we may use any of the well-recommended root-finding algorithm, e.g. Brent's method [15].

In our approximation the functions $h_\beta(\tau,\mu)$ and $g_\beta(\tau,\mu)$ (Eqs. (12) and (13)) may be written in the form

$$h_\beta(\tau,\mu) = 1 + \mu \sum_{i=1}^{N} \frac{a_i \exp(-s_i\tau)}{1+s_i\mu} \tag{24}$$

and

$$g_\beta(\tau,\mu) = \exp(-\tau/\mu) + \mu \sum_{i=1}^{N} \frac{a_i \left[\exp(-s_i\tau) - \exp(-\tau/\mu)\right]}{1-s_i\mu}. \tag{25}$$

It may happen that in certain cases we observe the apparent singularity at $s_i\mu = 1$ but it can simply be removed by substituting the respective term in the sum for $a_i\tau\exp(-\tau/\mu)$.

Since the formula for the emissive power at the boundary is given by Sobolev [18] in the form

$$B_\beta(0,\mu_0) = 1 + \int_0^\infty \Phi_\beta(\tau)\exp(-\tau/\mu_0)\mathrm{d}\tau, \tag{26}$$

we have in our approximation

$$B_\beta(0,\mu_0) = 1 + \mu \sum_{i=1}^{N} \frac{a_i}{1+s_i\mu_0}. \tag{27}$$

This concludes the solution of Eq. (8).

## 3 THE $H-$ FUNCTION

In this section we describe two more methods to determine the emissive power at the boundary with the intention to use them in estimation of the accuracy of the results for the emissive power at arbitrary optical depths obtained by our approximation method.

First, Breig and Crosbie have shown that the emissive power at the boundary satisfies the well-known Ambarzumian-Chandrasekhar non-linear integral equation for the $H-$ function in the form [1]

$$H_\beta(\mu) = 1 + \mu H(\mu) \int_0^p \frac{\Psi(x,\beta) H_\beta(x) \mathrm{d}x}{\mu + x}, \tag{28}$$

where the $H-$function is closely connected with the emissive power at the boundary,

$$H_\beta(\mu) = B_\beta(0,\mu), \tag{29}$$

or

$$h_\beta(0,\mu) = B_\beta(0,\mu). \tag{30}$$

Eq. (28) may be solved by successive approximation which is a standard iterative technique in solving integral equations but, according to Chandrasekhar [14] the equation must be modified in order to get a rapidly converging scheme. The modified equation has the form

$$H_\beta^{-1}(\mu) = \left[1 - \frac{1}{\beta} \arcsin \frac{\beta}{\sqrt{1+\beta^2}}\right]^{1/2} + \int_0^p \frac{x\Psi(x,\beta) H_\beta(x) \mathrm{d}t}{x+\mu}. \tag{31}$$

We approximate the integral in Eq. (31) by a finite sum using a Gaussian quadrature rule in the interval (0, $p$). Then we start the iteration process by taking $H_\beta(\mu) = 1$ as the zeroth approximation. The iteration process may be substantially accelerated if we take the average of two subsequent approximations as the next approximation.

Last but not least we may use the explicit solution of Eq. (28) given by Chandrasekhar [14]

$$\ln H_\beta(\mu) = \frac{\mu}{2\pi i} \int_{-i\infty}^{i\infty} \ln T(w,\beta) \frac{z \mathrm{d}w}{w^2 - \mu^2}, \tag{32}$$

where

$$T(w,\beta) = 1 - 2w^2 \int_0^p \frac{\Psi(\mu,\beta)\mathrm{d}\mu}{w^2-\mu^2}. \tag{33}$$

According to Kourganoff and Busbridge [16] the complex integral in Eq.(32) can be transformed into a real integral by the substitution $w = i\cot\vartheta$. Using this substitution in Eqs. (32) and (33) we obtain

$$T(\vartheta,\beta) = 1 - \frac{\cos\vartheta}{\sqrt{\beta^2+\sin^2\vartheta}} \arctan\frac{\sqrt{\beta^2+\sin^2\vartheta}}{\cos\vartheta} \tag{34}$$

and

$$\ln H_\beta(\mu) = -\frac{\mu}{\pi}\int_0^{\pi/2} \frac{\ln T(\vartheta,\beta)\mathrm{d}\vartheta}{\cos^2\vartheta + \mu^2\sin^2\vartheta}. \tag{35}$$

Stibbs and Weir [17] have calculated $H-$functions for isotropic scattering by direct quadrature of Eq. (35) after having carefully removed all the singularities of the integrand. We closely follow their example. For the problem on hand there is only one case which require special analyse and which follows from the fact that

$$\lim_{\mu\to 0} \partial H_\beta/\partial\mu \to \infty. \tag{36}$$

For small values of $\mu$ the integrand in Eq. (35) displays a sharp minimum when $\vartheta$ approaches $\pi/2$ and reaches zero at $\vartheta = \pi/2$. Stibbs and Weir eliminated this minimum by integrating by parts and so do we. They noticed that

$$\int \frac{\mathrm{d}\vartheta}{\cos^2\vartheta + \mu^2\sin^2\vartheta} = \mu^{-1}\arctan(\mu\tan\vartheta), \tag{37}$$

and the integrated part vanishes. If we write (cf. Eq.(35))

$$H_\beta(\mu) = \exp\left[I\left(\mu,\beta\right)\right], \tag{38}$$

we obtain

$$I(\mu,\beta) = \frac{1}{\pi}\int_0^{\pi/2} f(\vartheta)\mathrm{d}\vartheta, \tag{39}$$

where

$$f(\vartheta) = \left[\frac{\partial}{\partial\vartheta}\ln T(\vartheta,\beta)\right]\arctan(\mu\tan\vartheta) \tag{40}$$

and

$$\frac{\partial}{\partial\vartheta}\ln T(\vartheta,\beta)=\frac{1}{T(\vartheta,\beta)}\left[\frac{(1+\beta^2)\sin\vartheta}{\left(\beta^2+\sin^2\vartheta\right)^{3/2}}\arctan\frac{\sqrt{\beta^2+\sin^2\vartheta}}{\cos\vartheta}-\frac{\cos\vartheta\sin\vartheta}{\beta^2+\sin^2\vartheta}\right]. \tag{41}$$

This unwieldy function is nevertheless a well-behaved monotonic function of $\vartheta$ apart from the fact that its first derivative at $\vartheta=\pi/2$ is large when $\mu$ is small. We may eliminate this unpleasant feature by introducing a new function

$$g(\vartheta)=\frac{\pi}{2\sqrt{1+\beta^2}}\arctan(\mu\tan\vartheta), \tag{42}$$

and considering the function

$$I(\mu,\beta)=\frac{1}{\pi}\int_0^{\pi/2}[f(\vartheta)-g(\vartheta)\mathrm{d}\vartheta+\frac{1}{\pi}\int_0^{\pi/2}g(\vartheta)\mathrm{d}\vartheta=I_1(\mu,\beta)+I_2(\mu,\beta). \tag{43}$$

Now we can find the function $I_1(\mu,\beta)$ with reasonable accuracy using the Gaussian quadrature since the integrand is a well-behaved function. Stibbs and Weir [17] have shown that the second function may be evaluated by using the following series expansions

$$I_2(\mu,\beta)=\frac{1}{2\sqrt{1+\beta^2}}\left\{\frac{1}{2}\ln\mu\ln\frac{1-\mu}{1+\mu}+\sum_{n=0}^{\infty}\frac{\mu^{2n+1}}{(2n+1)^2}\right\},\qquad 0\le\mu<1 \tag{44}$$

or

$$=\frac{1}{2\sqrt{1+\beta^2}}\left\{\frac{\pi^2}{8}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(\frac{1-\mu}{1+\mu}\right)^{2n+1}\right\},\qquad 0<\mu\le 1, \tag{45}$$

while Eq.(44) should be used for $\mu\le\sqrt{2}-1$ and Eq. (45) for $\mu>\sqrt{2}-1$. Because of Eq. (36) the determination of the $H-$function at small values of $\mu$ still remains a problem and it would be desirable to have an approximation formula the accuracy of which would improve with $\mu\to 0$. This formula may simply be obtained by letting $H_\beta(\mu)=1$ in the right hand side of Eq. (28) and integrating it in a straightforward way. The result is

$$H_\beta(\mu)=1+\frac{\mu}{2\sqrt{1+\beta^2}}\ln\frac{2\sqrt{1+\beta^2}}{\mu\left(1+\sqrt{1+\beta^2}\right)}+O(\mu^2). \tag{46}$$

There is another critical region when $\mu\to\infty$. When using any of the three methods for determining the $H-$ function we observe a rapid deterioration of accuracy in this region. This difficulty may be overcome by writing the Eq. (18) for large values of $\mu$ in the form

$$H_\beta(\mu) = 1 + H(\mu)\left[h_0(\beta) - h_1(\beta)\mu^{-1} + h_2(\beta)\mu^{-2} - h_3(\beta)\mu^{-3} + ...\right], \quad (47)$$

where

$$h_n(\beta) = \int_0^1 \Psi(\mu,\beta) H_\beta(\mu)\mu^n \mathrm{d}\mu. \quad (48)$$

Hence, for large values of $\mu$ we may use the following formula

$$H_\beta(\mu) = \left[1 - \sum_{n=0}^{\infty} (-1)^n h_n(\beta)\mu^{-n}\right]^{-1}. \quad (49)$$

As a special case we have

$$H_\beta(\infty) = [1 - h_0(\beta)]^{-1}. \quad (50)$$

There is a problem, though, since for large values of parameter $\beta$ the characteristic function increases very rapidly if $\mu \to 1$ and we encounter a rapid loss in accuracy when using Eq. (48). This problem may be by-passed by using a well-recommended way - in Eq. (48) we subtract the maximum value of the integrand and, since this can be integrated analytically, add it later. We obtain as a result

$$h_n(\beta) = \int_0^1 \Psi(\mu,\beta)\left[H_\beta(\mu) - H_\beta(1)\right]\mu^n \mathrm{d}\mu + H_\beta(1)\psi_n(\beta), \quad (51)$$

where the $n$ -th moment of the characteristic function is as follows

$$\psi_n(\beta) = \int_0^p \Psi(\mu,\beta)\mu^n \mathrm{d}\mu. \quad (52)$$

For the first four moments of the characteristic function we obtain

$$\begin{aligned} \psi_0(\beta) &= \frac{1}{2\beta}\arcsin\frac{\beta}{\sqrt{1+\beta^2}}, \\ \psi_1(\beta) &= \frac{1}{2\beta^2}\left(1 - \frac{1}{\sqrt{1+\beta^2}}\right), \\ \psi_2(\beta) &= \frac{1}{4\beta^2}\left(-\frac{1}{1+\beta^2} + \frac{1}{\beta}\arcsin\frac{\beta}{\sqrt{1+\beta^2}}\right), \\ \psi_3(\beta) &= \frac{1}{6\beta^2}\left[\frac{2}{\beta^2} - \left(\frac{2}{\beta^2} + \frac{1}{1+\beta^2}\right)\frac{1}{\sqrt{1+\beta^2}}\right]. \end{aligned} \quad (53)$$

For evaluating these integrals we have used the free service provided by the Wolfram Research, Inc. at the web-page http://www.integrals.com/index.cgi.

# 4 RADIATIVE FLUX

In this section we consider the formulation of the equations for the $z-$component of radiative flux in the atmosphere and respective calculations. According to [10] the $z-$component of radiative flux can be shown to satisfy the relationship

$$q_z(\tau_y, \tau) = I_0 Q_{\beta=0}(\tau, \mu_0) + \epsilon I_0 Q_\beta(\tau, \mu_0) \exp(i\beta\tau_y), \tag{54}$$

where the dimensionless radiative flux is given by

$$Q_\beta(\tau, \mu_0) = \mu_0 \exp(-\tau/\mu_0) + \frac{1}{2}\int_0^\tau \mathcal{E}_2(\tau - \tau', \beta) B_\beta(\tau', \mu_0) \mathrm{d}\tau' - \frac{1}{2}\int_\tau^\infty \mathcal{E}_2(\tau' - \tau, \beta) B_\beta(\tau', \mu_0) \mathrm{d}\tau'. \tag{55}$$

In Eq. (55) the generalized second exponential integral can be written as

$$\mathcal{E}_2(\tau, \beta) = \int_0^p \exp(-|\tau|/u) \frac{\mathrm{d}u}{(1-\beta^2u^2)^{3/2}}. \tag{56}$$

Substituting Eq. (56) into Eq. (55), changing the order of integration and taking into account Eqs. (2), (3), (14) and (15) we obtain

$$Q_\beta(\tau, \mu_0) = \mu_0 \exp(-\tau/\mu_0) + \mu_0 H_\beta(\mu_0) \int_0^p \frac{u\psi_1(\mu,\beta)\mathrm{d}u}{\mu_0 - u}\left[g_\beta(\tau, \mu_0) - g_\beta(\tau, u)\right] - \mu_0 H_\beta(\mu_0) \int_0^p \frac{u\psi_1(\mu,\beta)\mathrm{d}u}{\mu_0 + u}\left[g_\beta(\tau, \mu_0) + h_\beta(\tau, u) - 1\right], \tag{57}$$

where

$$\psi_1(\mu, \beta) = \frac{1}{2(1-\beta^2u^2)^{3/2}}. \tag{58}$$

The radiative flux at the boundary of an atmosphere is thus

$$Q_\beta(0, \mu_0) = \mu_0 - \mu_0 H_\beta(\mu_0) \int_0^p \frac{u\psi_1(\mu,\beta)H_\beta(u)\mathrm{d}u}{\mu_0 + u}. \tag{59}$$

As far as the problem under consideration formally coincides with the respective problem of the one-dimensional transfer with conservative scattering we may expect the total radiative flux to disappear if $\beta = 0$. This happens because in a conservative atmosphere photons are not destroyed and because the atmosphere is semi-infinite all the photons incident on such an atmosphere must emerge through the plane of incidence thus causing the total flux of energy to disappear. This is why in Eq. (54) we are left only with the part of the flux for which $\beta \neq 0$. At the same time this means that because of the cosine function there occurs the change of sign of the flux which means that the energy flow changes its direction to the opposite at

$$\tau_y = \frac{(2n+1)\pi}{2\beta}, \qquad n = 0, 1, 2, ...$$

# 5 NUMERICAL RESULTS

Due to the facts that for large $\beta$ the characteristic function $\Psi(\mu, \beta)$ increases rapidly if $\mu \to p$ and that the generalized exponential integral has a logarithmic singularity at $\tau \to 0$ we have to be careful in using the Gaussian rule in Eq. (19). For better accuracy we divided the integration range $(0, p)$ into four subintervals $(0, 0.1p)$, $(0.1p, 0.9p)$, $(0.9p, 0.99p)$ and $(0.99p, p)$ and in each subinterval we used the Gaussian rule with $N/4$ points. We admit that this division is arbitrary and perhaps a better scheme at the same computational volume could be found, but as far as this scheme at $N = 84$ secured at least five significant figures both for the dimensionless flux and the emissive power in the region of $0 \leq \beta \leq 10000$, we remained content with the scheme described. Instead of a rigorous error estimation we firstly compared our results with those by Breig and Crosbie [7], which were obtained at the boundary of the atmosphere only, and found the coincidence to be good. And secondly, we gradually increased the number of quadrature points and compared the respective results in large number of numerical experiments.

Fig. 1 shows the behaviour of the dimensionless emissive power for $\mu = 1.0$. According to calculations for small angles of incidence the dimensionless emissive power decreases monotonically for all values of $\beta$. This is not the case for the perpendicular incidence where such a monotonous decrease is present only for $\log \beta \geq 0.5$.For smaller values of $\beta$ the dimensionless emissive power increases with the optical depth $\tau$ till it reaches a maximum and only then starts to decrease.

The influence of the spatial frequency $\beta$ and the optical depth $\tau$ (optical coordinate in $z-$ direction) on the dimensionless flux of energy $Q_\beta(\tau, \mu = 1)$ is illustrated in Fig.2. The dimensionless flux decreases from its maximum value (which is equal to $\mu$ according to Eq. (55)) at large values of spatial frequency and small values of optical depth towards smaller values of spatial frequency and larger values of optical depth but curiously enough there appears a maximum in the $Q_\beta$ - $\log \tau$ plane in the region where $\log \tau \geq -0.75$.

Fig. 3 displays the flux as a function of optical coordinates $\tau_y$ and $\tau$ for parameters $\beta = 100.0$ and $\mu = 1.0$. We may observe the monotonous decrease of the flux towards larger values of $\tau$ till it becomes zero at $\log \tau \sim 0.75$. At the boundary the flux is determined by the incident radiation and it changes direction as predicted.

The surface of the emissive power (or the temperature distribution) as a function of optical coordinates $\tau_y$ and $\tau$ for parameters $\beta = 0.01$ and $\mu = 1.0$ is presented in Fig. 4. In this case the influence of the incident radiation disappears very deep in the atmosphere, or, in other words, the emissive power reaches its undisturbed state at $\log \tau \sim 3$ only. We may recall the presence of the maxima in the run of the dimensionless emissive power which now causes maxima also in the (total) emissive power. The increase in the spatial frequency causes the emissive power to reach the undisturbed state at much smaller optical depths, e.g. at $\beta = 100.0$ the respective optical depth is only $\log \tau \sim 0.5$ (Fig. 5).

When considering the oblique incidence on the atmosphere we may observe that the optical depth of undisturbed state is still defined by the spatial frequency, only the value of undisturbed emissive power is smaller at smaller angles of incidence (Fig. 6).

# 6 CONCLUSION

It has been shown that the techniques used in one-dimensional radiative transfer, namely the approximation of the Sobolev resolvent function by a sum of exponents can be freely used for certain problems of two-dimensional radiative transfer. This aproximation is simple and straightforward, giving at the same time accurate and reliable results.

# 7 ACKNOWLEDGEMENTS

This work has been supported by the Estonian Ministry of Education within Project No. TO 0060059S98 and by the Estonian Scientific Council under Grant No. 2629.

I would like to express my gratitude to Prof. A.L. Crosbie for sending me some offprints of his papers.

**References**


1. Chandrasekhar, S. On the diffuse reflection of a pencil of radiation by a plane parallel atmosphere. *Proc. Nat. Acad. Sci. U.S.A.*, 1958, **44**, 933-940.

2. Bellman, R., Kalaba, R. and Ueno, S. On the diffuse reflection of parallel rays by an inhomogeneous flat layer as a limiting process. *J. Math. Anal. Appl.*, 1963, **7**.

3. Bellman, R., Kalaba, R. and Ueno, S. Invariant imbedding and the time dependent diffuse reflection of a pencil of radiation by a finite inhomogeneous flat layer -I . *J. Math. Anal. Appl.*, 1963, **7,** 310-321.

4. Bellman, R., Kalaba, R. and Ueno, S. Invariant imbedding and diffuse reflection from a two-dimensional flat layer. *Icarus*, 1963, **1**, 297-303.

5. Rybicki, G.B. The searchlight problem with isotropic scattering. *J. Quant. Spectrosc. Radiat. Tranfer*, 1971, **11**, 827-849.

6. Breig, W.F. and Crosbie, A.L. Two-dimensional radiative equilibrium. *J. Math. Anal. Appl., 1974,* **46***, 104-125.*

7. Breig, W.F. and Crosbie, A.L. Two-dimensional radiative equilibrium: A semi-infinite medium subjected to cosine varying radiation. *J. Quant. Spectrosc. Radiat. Transfer*, 1973, **13**, 1395-1419.

8. Breig, W.F. and Crosbie, A.L. Two-dimensional radiative equilibrium: A semi-infinite medium subjected to a finite strip of radiation. *J. Quant. Spectrosc. Radiat. Transfer*, 1974, **14**, 189-209.

9. Breig, W.F. and Crosbie, A.L. Two-dimensional radiative equilibrium: Boundary emissive powers for a finite medium subjected to cosine varying radiation. *J. Quant. Spectrosc. Radiat. Transfer*, 1974, **14**, 1209-1237.

10. Breig, W.F. and Crosbie, A.L. Two-dimensional radiative equilibrium: Boundary fluxes for a finite medium subjected to cosine varying radiation. *J. Quant. Spectrosc. Radiat. Transfer*, 1975, **15**, 163-179.

11. Breig, W.F. and Crosbie, A.L. Numerical computation of a generalized exponential integral function. *Math. Comp.*, 1974, **28**, 575-579.

12. Mueller, D.W. and Crosbie, A.L. Three-dimensional radiative transfer with polarization in a multiple scattering medium exposed to spatially varying radiation. *J. Quant. Spectrosc. Radiat. Transfer*, 1997, **57**, 81-105.

13. Viik, T., Rōōm, R. and Heinlo, A. A method of the resolvent function approximation in radiative transfer. *Tartu Teated*, 1985, No. 76, 3-131. (In Russian)

14. Chandrasekhar, S. *Radiative Transfer.* Oxford University Press, 1950. [Dover, 1960].

15. Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T. *Numerical recipes*, Cambridge University Press, Cambridge, 1986.

16. Kourganoff, V. and Busbridge, I.W. *Basic Methods in Transfer Problems.* Clarendon Press, Oxford, 1952.

17. Stibbs, D.W.N. and Wier, R.E. On the H-functions for isotropic scattering. *Monthly Notices of the Royal Astron. Soc.*, 1959, 119, 512-525.

18. Sobolev, V.V. *Light Scattering in Planetary Atmospheres.* Nauka, Moscow, 1972. (In Russian)

Temperatuurijaotus pool-lõpmatus atmosfääris, millele langeb kosiinus-seaduse järgi muutuv kollimeeritud kiirgus

Tõnu VIIK

Vaatleme kiirguslevi pool-lõpmatu optilise paksusega kahemõõtmelises tasaparalleelses mittehajutavas, kuid neelavas ja kiirgavas atmosfääris, millele langeb kosiinus-seaduse järgi muutuv kollimeeritud kiirgus. Oletame veel, et atmosfäär on hall ja ta on kiirguslikus tasakaalus, s.t. energia levib seal vaid kiirguse teel. Kiirguslevi võrrandi saab taandada integraalvõrrandiks, mille omakorda saab muutujate eraldamise teel taandada suhteliselt lihtsaks ühemõõtmeliseks integraalvõrrandiks. See võrrand erineb tavalise ühemõõtmelise kiirguslevi võrrandist karakteristliku funktsiooni poolest, mis pole enam paarisfunktsiooniline polünoom, vaid palju keerulisem, kuid siiski paarsuse säilitanud funktsioon. Osutub, et selle integraalvõrrandi lahendamiseks saab kasutada meetodit, kus integraalvõrrandi tuum lähendatakse eksponentide reaga. Sellisel juhul lahendub lähendvõrrand täpselt, kusjuures lahendiks on samuti eksponentide rida, mille koefitsiendid saab lihtsatest võrranditest leida. Edasi defineerime funktsioonid $h$ ja $g$ nagu ühemõõtmelisel juhulgi ning sellega on kiirgusväli ülalkirjeldatud atmosfääri igas punktis leitud.

Eraldi uurime võimalusi $H$-funktsiooni numbriliseks leidmiseks, sest see on funktsioon, mis määrab kiirgusvõime (või temperatuurijaotuse) väärtuse atmosfääri pinnal. Kaht $H$-funktsiooni numbrilise leidmise meetodit kasutasime oma lahendusmeetodiga saadud tulemuste kontrolliks ja veendusime, et meie meetod annab väga täpseid tulemusi. Numbrilised eksperimendid parameetrite erinevate väärtuste puhul näitasid, et langeva kiirguse ruumilise sageduse teatud väärtuste puhul võib kiirgusvoog mingitel optilistel sügavustel omada maksimumi ja kollimeeritud "triipude" vahel muuta isegi suunda. Samuti selgus, et temperatuuri jaotusfunktsioon võib konstantse väärtuseni jõuda alles väga sügaval atmosfääri sees, kus optiline sügavus on suurusjärgus 1000.

## 7.1 Figure captions

Fig.1. The dimensionless emissive power as a function of optical coordinate $\tau$ and spatial frequency $\beta$ at angle of incidence $0°$.

Fig.2. The dimensionless flux as a function of spatial frequency $\beta$ and optical coordinate $\tau$ at angle of incidence $0°$.

Fig.3. The flux as a function of optical coordinates $\tau$ and $\tau_y$ at spatial frequency $\beta = 100.0$ and at angle of incidence $72°.54$.

Fig.4. The emissive power as a function of optical coordinates $\tau$ and $\tau_y$ at spatial frequency $\beta = 0.01$ and at angle of incidence $0°$.

Fig.5. The emissive power as a function of optical coordinates $\tau$ and $\tau_y$ at spatial frequency $\beta = 100.0$ and at angle of incidence $0°$.

Fig. 6. The emissive power as a function of optical coordinates $\tau$ and $\tau_y$ at spatial frequency $\beta = 0.01$ and at angle of incidence $72°.54$.